Exercise Solutions for *Introduction to 3D Game Programming with DirectX 10*

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Solutions to Part I

Chapter 2

1. Solve the following matrix equation for $\mathbf{X}$: $3\left(\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} - 2\mathbf{X}\right) = 2\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$.

Solution:

$$\begin{aligned}
3\left(\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} - 2\mathbf{X}\right) &= 2\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \\
\begin{bmatrix} -6 & 0 \\ 3 & 9 \end{bmatrix} - 6\mathbf{X} &= \begin{bmatrix} -4 & 0 \\ 2 & 6 \end{bmatrix} \\
-6\mathbf{X} &= \begin{bmatrix} -4 & 0 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} -6 & 0 \\ 3 & 9 \end{bmatrix} \\
-6\mathbf{X} &= \begin{bmatrix} 2 & 0 \\ -1 & -3 \end{bmatrix} \\
\mathbf{X} &= \begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

2. Compute the following matrix products:

a) $\begin{bmatrix} -2 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 6 \\ 2 & -3 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solution:

$$\begin{aligned}
\begin{bmatrix} -2 & 0 & 3 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 6 \\ 2 & -3 \end{bmatrix} &= \begin{bmatrix} (-2)(2) + 0(0) + 3(2) & (-2)(-1) + 0(6) + 3(-3) \\ 4(2) + 1(0) + (-1)(2) & 4(-1) + 1(6) + (-1)(-3) \end{bmatrix} \\
&= \begin{bmatrix} 2 & -7 \\ 6 & 5 \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}-2 & 0\\1 & 1\end{bmatrix}=\begin{bmatrix}1(-2)+2(1) & 1(0)+2(1)\\3(-2)+4(1) & 3(0)+4(1)\end{bmatrix}=\begin{bmatrix}0 & 2\\-2 & 4\end{bmatrix}$$

$$\begin{bmatrix}2 & 0 & 2\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}2(1)+0(2)+2(1)\\0(1)+(-1)(2)+(-3)(1)\\0(1)+0(2)+1(1)\end{bmatrix}=\begin{bmatrix}4\\-5\\1\end{bmatrix}$$

3. Compute the transpose of the following matrices:

a) $[1,\quad 2,\quad 3]$, b) $\begin{bmatrix}x & y\\z & w\end{bmatrix}$, c) $\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\\7 & 8\end{bmatrix}$

Solution:

$$[1,\quad 2,\quad 3]^T=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}x & y\\z & w\end{bmatrix}^T=\begin{bmatrix}x & z\\y & w\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\\7 & 8\end{bmatrix}^T=\begin{bmatrix}1 & 3 & 5 & 7\\2 & 4 & 6 & 8\end{bmatrix}$$

4. Let $=\begin{bmatrix}2 & 0 & 1\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}$. Is $\mathbf{B}=\begin{bmatrix}1/2 & 0 & -1/2\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}$ the inverse of $\mathbf{A}$?

Solution:

$$\mathbf{AB}=\begin{bmatrix}2 & 0 & 1\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1/2 & 0 & -1/2\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\mathbf{I}$$

$$\mathbf{BA}=\begin{bmatrix}1/2 & 0 & -1/2\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\0 & -1 & -3\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}=\mathbf{I}$$

Therefore, $\mathbf{B}$ is the inverse of $\mathbf{A}$.

5. Let $\mathbf{A}=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$. Is $\mathbf{B}=\begin{bmatrix}-2 & 1\\3/2 & 1/2\end{bmatrix}$ the inverse of $\mathbf{A}$?

Solution:

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

Since $\mathbf{AB} \neq \mathbf{I}$ we can conclude that $\mathbf{B}$ is not the inverse of $\mathbf{A}$.

6. Write the following linear combinations as vector-matrix products:

a) $\mathbf{v} = 2(1,2,3) - 4(-5,0,-1) + 3(2,-2,3)$
b) $\mathbf{v} = 3(2,-4) + 2(1,4) - 1(-2,-3) + 5(1,1)$

Solution:

Apply Equation 2.3:

$$\mathbf{v} = [2,-4,3] = \begin{bmatrix} 1 & 2 & 3 \\ -5 & 0 & -1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\mathbf{v} = [3,2,-1,5] \begin{bmatrix} 2 & -4 \\ 1 & 4 \\ -2 & -3 \\ 1 & 1 \end{bmatrix}$$

7. Show that

$$\mathbf{AB} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} = \begin{bmatrix} \leftarrow \mathbf{A}_{1,*}\mathbf{B} \longrightarrow \\ \leftarrow \mathbf{A}_{2,*}\mathbf{B} \longrightarrow \\ \leftarrow \mathbf{A}_{3,*}\mathbf{B} \longrightarrow \end{bmatrix}$$

Solution:

Let i be an arbitrary row in $\mathbf{AB}$. By definition of matrix multiplication, we know that the ith row is given by

$$(\mathbf{AB})_{i,*} = [\mathbf{A}_{i,*} \cdot \mathbf{B}_{*,1} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,2} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,3}]$$

However, by definition of matrix multiplication, this is equal to the vector-matrix product $\mathbf{A}_{i,*}\mathbf{B}$. That is,

$$(\mathbf{AB})_{i,*} = [\mathbf{A}_{i,*} \cdot \mathbf{B}_{*,1} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,2} \quad \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,3}] = \mathbf{A}_{i,*}\mathbf{B}$$

Since i was an arbitrary row, we just substitute $i = 1,2,3$ to complete the proof:

$$\mathbf{AB} = \begin{bmatrix} (\mathbf{AB})_{1,*} \\ (\mathbf{AB})_{2,*} \\ (\mathbf{AB})_{3,*} \end{bmatrix} = \begin{bmatrix} \leftarrow \mathbf{A}_{1,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{2,*}\mathbf{B} \rightarrow \\ \leftarrow \mathbf{A}_{3,*}\mathbf{B} \rightarrow \end{bmatrix}$$

8. Show that

$$\mathbf{Au} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{A}_{*,1} + y\mathbf{A}_{*,2} + z\mathbf{A}_{*,3}$$

Solution:

You can take a brute force approach and follow the same pattern used to derive Equation 2.2. However, we will show another strategy so that we can demonstrate some additional transpose properties, which will also be useful for Exercise 9. We need to prove the following three properties about the transpose:

1. $(\mathbf{A}^T)^T = \mathbf{A}$
2. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
3. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

Property 1. The first property is trivial. Consider an arbitrary element A_{ij}. Then $(\mathbf{A}^T)_{ij} = A_{ji}$. Finally, $((\mathbf{A}^T)^T)_{ij} = (\mathbf{A}^T)_{ji} = A_{ij}$ for all ij. So $(\mathbf{A}^T)^T = \mathbf{A}$.

Property 2. For Assume $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix. We show that the elements of the left-hand-side equal the elements of the right-hand-side, and therefore, the matrices are equal. Consider the *ij*th element in $\mathbf{AB}$; it has the form:

$$(\mathbf{AB})_{ij} = \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,j}$$

By definition of the transpose, we have:

$$((\mathbf{AB})^T)_{ji} = \mathbf{A}_{i,*} \cdot \mathbf{B}_{*,j}$$

Thus,

$$((\mathbf{AB})^T)_{ij} = \mathbf{A}_{j,*} \cdot \mathbf{B}_{*,i}$$

Now, consider the *ij*th elements in $\mathbf{A}$ and $\mathbf{B}$, which are A_{ij} and B_{ij}, respectively. We have,

$$(\mathbf{B}^T\mathbf{A}^T)_{ij} = (\mathbf{B}^T)_{i,*} \cdot (\mathbf{A}^T)_{*,j} = \mathbf{B}_{*,i} \cdot \mathbf{A}_{j,*} = \mathbf{A}_{j,*} \cdot \mathbf{B}_{*,i}$$

Consequently, we have that $(\mathbf{AB})^T_{ij} = (\mathbf{B}^T\mathbf{A}^T)_{ij}$ for all ij. Therefore, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Property 3. We show that the elements of the left-hand-side equal the elements of the right-hand-side, and therefore, the matrices are equal. Consider the *ij*th element in $\mathbf{A}+\mathbf{B}$; it has the form:

$$(\mathbf{A}+\mathbf{B})_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$$

By definition of the transpose, we have:

$$((\mathbf{A}+\mathbf{B})^T)_{ji} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$$

Thus,

$$((\mathbf{A}+\mathbf{B})^T)_{ij} = \mathbf{A}_{ji} + \mathbf{B}_{ji}$$

Now, consider the *ij*th elements in $\mathbf{A}$ and $\mathbf{B}$, which are A_{ij} and B_{ij}, respectively. We have,

$$(\mathbf{A}^T + \mathbf{B}^T)_{ij} = (\mathbf{A}^T)_{ij} + (\mathbf{B}^T)_{ij} = \mathbf{A}_{ji} + \mathbf{B}_{ji}$$

Consequently, we have that $((\mathbf{A}+\mathbf{B})^T)_{ij} = (\mathbf{A}^T + \mathbf{B}^T)_{ij}$ for all ij. Therefore, $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

With those three properties, we can recast the problem into something we already know about vector-matrix multiplication. We have:

$$\begin{aligned}
\mathbf{Au} &= ((\mathbf{Au})^T)^T \\
&= (\mathbf{u}^T\mathbf{A}^T)^T \\
&= ([x, y, z]\mathbf{A}^T)^T \\
&= \left(x(\mathbf{A}^T)_{1,*} + y(\mathbf{A}^T)_{2,*} + z(\mathbf{A}^T)_{3,*}\right)^T \\
&= x\left((\mathbf{A}^T)_{1,*}\right)^T + y\left((\mathbf{A}^T)_{2,*}\right)^T + z\left((\mathbf{A}^T)_{3,*}\right)^T
\end{aligned}$$

But row i of $\mathbf{A}^T$ is just column i of $\mathbf{A}$. So,

$$\begin{aligned}
&= x\left((\mathbf{A}^T)_{1,*}\right)^T + y\left((\mathbf{A}^T)_{2,*}\right)^T + z\left((\mathbf{A}^T)_{3,*}\right)^T \\
&= x\mathbf{A}_{*,1} + y\mathbf{A}_{*,2} + z\mathbf{A}_{*,3}
\end{aligned}$$

9. Show that $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$, assuming $\mathbf{A}$ is invertible.

Solution:

$$\begin{aligned}
\mathbf{A}^T(\mathbf{A}^{-1})^T &= (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T = \mathbf{I} \\
(\mathbf{A}^{-1})^T\mathbf{A}^T &= (\mathbf{A}\mathbf{A}^{-1})^T = \mathbf{I}^T = \mathbf{I}
\end{aligned}$$

Therefore, $(\mathbf{A}^{-1})^T$ is the inverse of $\mathbf{A}^T$.

10. Let $\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. Show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$. This shows that matrix multiplication is associative for 2×2 matrices. (In fact, matrix multiplication is associative for general sized matrices, whenever the multiplication is defined.)

Solution:

For 2×2 matrices, we will just do the computations:

$$\mathbf{BC} = \begin{bmatrix} B_{11}C_{11} + B_{12}C_{21} & B_{11}C_{12} + B_{12}C_{22} \\ B_{21}C_{11} + B_{22}C_{21} & B_{21}C_{12} + B_{22}C_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} A_{11}(B_{11}C_{11} + B_{12}C_{21}) + A_{12}(B_{21}C_{11} + B_{22}C_{21}) & A_{11}(B_{11}C_{12} + B_{12}C_{22}) + A_{12}(B_{21}C_{12} + B_{22}C_{22}) \\ A_{21}(B_{11}C_{11} + B_{12}C_{21}) + A_{22}(B_{21}C_{11} + B_{22}C_{21}) & A_{21}(B_{11}C_{12} + B_{12}C_{22}) + A_{22}(B_{21}C_{12} + B_{22}C_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}C_{11} + A_{11}B_{12}C_{21} + A_{12}B_{21}C_{11} + A_{12}B_{22}C_{21} & A_{11}B_{11}C_{12} + A_{11}B_{12}C_{22} + A_{12}B_{21}C_{12} + A_{12}B_{22}C_{22} \\ A_{21}B_{11}C_{11} + A_{21}B_{12}C_{21} + A_{22}B_{21}C_{11} + A_{22}B_{22}C_{21} & A_{21}B_{11}C_{12} + A_{21}B_{12}C_{22} + A_{22}B_{21}C_{12} + A_{22}B_{22}C_{22} \end{bmatrix}$$

$$(\mathbf{AB})\mathbf{C} = \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21})C_{11} + (A_{11}B_{12} + A_{12}B_{22})C_{21} & (A_{11}B_{11} + A_{12}B_{21})C_{12} + (A_{11}B_{12} + A_{12}B_{22})C_{22} \\ (A_{21}B_{11} + A_{22}B_{21})C_{11} + (A_{21}B_{12} + A_{22}B_{22})C_{21} & (A_{21}B_{11} + A_{22}B_{21})C_{12} + (A_{21}B_{12} + A_{22}B_{22})C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11}C_{11} + A_{12}B_{21}C_{11} + A_{11}B_{12}C_{21} + A_{12}B_{22}C_{21} & A_{11}B_{11}C_{12} + A_{12}B_{21}C_{12} + A_{11}B_{12}C_{22} + A_{12}B_{22}C_{22} \\ A_{21}B_{11}C_{11} + A_{22}B_{21}C_{11} + A_{21}B_{12}C_{21} + A_{22}B_{22}C_{21} & A_{21}B_{11}C_{12} + A_{22}B_{21}C_{12} + A_{21}B_{12}C_{22} + A_{22}B_{22}C_{22} \end{bmatrix}$$

Comparing the terms element-by-element, we see $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.