Exercise Solutions for *Introduction to 3D Game Programming with DirectX 10*

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Solutions to Part I

Chapter 1

1. Let $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (3, -4)$. Perform the following computations and draw the vectors relative to a 2D coordinate system.

a) $\mathbf{u} + \mathbf{v}$
b) $\mathbf{u} - \mathbf{v}$
c) $2\mathbf{u} + \frac{1}{2}\mathbf{v}$
d) $-2\mathbf{u} + \mathbf{v}$

Solution:

a) $(1, 2) + (3, -4) = \big(1 + 3, 2 + (-4)\big) = (4, -2)$
b) $(1, 2) - (3, -4) = (1, 2) + (-3, 4) = (1 - 3, 2 + 4) = (-2, 6)$
c) $2(1, 2) + \frac{1}{2}(3, -4) = (2, 4) + \left(\frac{3}{2}, -2\right) = \left(\frac{7}{2}, 2\right)$
d) $-2(1, 2) + (3, -4) = (-2, -4) + (3, -4) = (1, -8)$

2. Let $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (3, -4, 1)$. Perform the following computations.

a) $\mathbf{u} + \mathbf{v}$
b) $\mathbf{u} - \mathbf{v}$
c) $3\mathbf{u} + 2\mathbf{v}$
d) $-2\mathbf{u} + \mathbf{v}$

Solution:

a) $(-1, 3, 2) + (3, -4, 1) = (-1 + 3, 3 + (-4), 2 + 1) = (2, -1, 3)$
b) $(-1, 3, 2) - (3, -4, 1) = (-1, 3, 2) + (-3, 4, -1) = (-4, 7, 1)$
c) $3(-1, 3, 2) + 2(3, -4, 1) = (-3, 9, 6) + (6, -8, 2) = (3, 1, 8)$
d) $-2(-1, 3, 2) + (3, -4, 1) = (2, -6, -4) + (3, -4, 1) = (5, -10, -3)$

3. This exercise shows that vector algebra shares many of the nice properties of real numbers (this is not an exhaustive list). Assume $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$. Also assume that c and k are scalars. Prove the following vector properties.

a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative Property of Addition)
b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (Associative Property of Addition)
c) $(ck)\mathbf{u} = c(k\mathbf{u})$ (Associative Property of Scalar Multiplication)
d) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ (Distributive Property 1)
e) $\mathbf{u}(k + c) = k\mathbf{u} + c\mathbf{u}$ (Distributive Property 2)

Solution:

a)

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_x, u_y, u_z) + (v_x, v_y, v_z) \\
&= (u_x + v_x, u_y + v_y, u_z + v_z) \\
&= (v_x + u_x, v_y + u_y, v_z + u_z) \\
&= (v_x, v_y, v_z) + (u_x, u_y, u_z) \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

b)

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_x, u_y, u_z) + \left((v_x, v_y, v_z) + (w_x, w_y, w_z)\right) \\
&= (u_x, u_y, u_z) + (v_x + w_x, v_y + w_y, v_z + w_z) \\
&= \left(u_x + (v_x + w_x), u_y + (v_y + w_y), u_z + (v_z + w_z)\right) \\
&= \left((u_x + v_x) + w_x, (u_y + v_y) + w_y, (u_z + v_z) + w_z\right) \\
&= (u_x + v_x, u_y + v_y, u_z + v_z) + (w_x, w_y, w_z) \\
&= \left((u_x, u_y, u_z) + (v_x, v_y, v_z)\right) + (w_x, w_y, w_z) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}$$

c)

$$\begin{aligned}
(ck)\mathbf{u} &= (ck)(u_x, u_y, u_z) \\
&= \left((ck)u_x, (ck)u_y, (ck)u_z\right) \\
&= \left(c(ku_x), c(ku_y), c(ku_z)\right) \\
&= c(ku_x, ku_y, ku_z) \\
&= c(k\mathbf{u})
\end{aligned}$$

d)

$$\begin{aligned}
k(\mathbf{u} + \mathbf{v}) &= k\left((u_x, u_y, u_z) + (v_x, v_y, v_z)\right) \\
&= k(u_x + v_x, u_y + v_y, u_z + v_z)
\end{aligned}$$

$$
\begin{aligned}
&= \left(k(u_x + v_x), k\left(u_y + v_y\right), k(u_z + v_z)\right) \\
&= \left(ku_x + kv_x, ku_y + kv_y, ku_z + kv_z\right) \\
&= \left(ku_x, ku_y, ku_z\right) + \left(kv_x, kv_y, kv_z\right) \\
&= k\mathbf{u} + k\mathbf{v}
\end{aligned}
$$

e)

$$
\begin{aligned}
\mathbf{u}(k + c) &= \left(u_x, u_y, u_z\right)(k + c) \\
&= \left(u_x(k + c), u_y(k + c), u_z(k + c)\right) \\
&= \left(ku_x + cu_x, ku_y + cu_y, ku_z + cu_z\right) \\
&= \left(ku_x, ku_y, ku_z\right) + \left(cu_x, cu_y, cu_z\right) \\
&= k\mathbf{u} + c\mathbf{u}
\end{aligned}
$$

4. Solve the equation $2\left((1, 2, 3) - \mathbf{x}\right) - (-2, 0, 4) = -2(1, 2, 3)$ for $\mathbf{x}$.

Solution: Use vector algebra to solve for $\mathbf{x}$:

$$
\begin{aligned}
2\left((1, 2, 3) - \mathbf{x}\right) - (-2, 0, 4) &= -2(1, 2, 3) \\
(2, 4, 6) - 2\mathbf{x} + (2, 0, -4) &= (-2, -4, -6) \\
-2\mathbf{x} + (2, 0, -4) &= (-4, -8, -12) \\
-2\mathbf{x} &= (-6, -8, -8) \\
\mathbf{x} &= (3, 4, 4)
\end{aligned}
$$

5. Let $\mathbf{u} = (-1, 3, 2)$ and $\mathbf{v} = (3, -4, 1)$. Normalize $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$
\|\mathbf{u}\| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}
$$

$$
\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \left(-\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)
$$

$$
\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2 + 1^2} = \sqrt{9 + 16 + 1} = \sqrt{26}
$$

$$
\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{3}{\sqrt{26}}, -\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)
$$

6. Let k be a scalar and let $\mathbf{u} = \left(u_x, u_y, u_z\right)$. Prove that $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$.

$$
\|k\mathbf{u}\| = \sqrt{(ku_x)^2 + \left(ku_y\right)^2 + (ku_z)^2} = \sqrt{k^2\left(u_x^2 + u_y^2 + u_z^2\right)} = |k|\sqrt{u_x^2 + u_y^2 + u_z^2} = |k|\|\mathbf{u}\|
$$

7. Is the angle between $\mathbf{u}$ and $\mathbf{v}$ orthogonal, acute, or obtuse?

a) $\mathbf{u} = (1,1,1), \mathbf{v} = (2,3,4)$
b) $\mathbf{u} = (1,1,0), \mathbf{v} = (-2,2,0)$
c) $\mathbf{u} = (-1,-1,-1), \mathbf{v} = (3,1,0)$

a) $\mathbf{u} \cdot \mathbf{v} = 1(2) + 1(3) + 1(4) = 9 > 0 \Rightarrow \text{acute}$
b) $\mathbf{u} \cdot \mathbf{v} = 1(-2) + 1(2) + 0(0) = 0 \Rightarrow \text{orthogonal}$
c) $\mathbf{u} \cdot \mathbf{v} = -1(3) + (-1)(1) + (-1)(0) = -4 < 0 \Rightarrow \text{obtuse}$

8. Let $\mathbf{u} = (-1,3,2)$ and $\mathbf{v} = (3,-4,1)$. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

Solution: Using the equation $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$ we have:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)\\ &= \cos^{-1}\left(\frac{-1(3) + 3(-4) + 2(1)}{\sqrt{14}\sqrt{26}}\right)\\ &= \cos^{-1}\left(\frac{-13}{\sqrt{14}\sqrt{26}}\right)\\ &= 132.95^\circ\end{aligned}$$

9. Let $\mathbf{u} = (u_x, u_y, u_z)$, $\mathbf{v} = (v_x, v_y, v_z)$, and $\mathbf{w} = (w_x, w_y, w_z)$. Also let c and k be scalars. Prove the following dot product properties.

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$
d) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
e) $\mathbf{0} \cdot \mathbf{v} = 0$

Solution:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (u_x, u_y, u_z) \cdot (v_x, v_y, v_z)\\ &= u_x v_x + u_y v_y + u_z v_z\\ &= (v_x, v_y, v_z) \cdot (u_x, u_y, u_z)\\ &= \mathbf{v} \cdot \mathbf{u}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (u_x, u_y, u_z) \cdot (v_x + w_x, v_y + w_y, v_z + w_z)\\ &= u_x(v_x + w_x) + u_y(v_y + w_y) + u_z(v_z + w_z)\\ &= u_x v_x + u_x w_x + u_y v_y + u_y w_y + u_z v_z + u_z w_z\\ &= (u_x v_x + u_y v_y + u_z) + (u_x w_x + u_y w_y + u_z w_z)\\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}\end{aligned}$$

$$\begin{aligned}k(\mathbf{u} \cdot \mathbf{v}) &= k(u_x v_x + u_y v_y + u_z v_z)\\ &= (ku_x)v_x + (ku_y)v_y + (ku_z)v_z\end{aligned}$$

$$
\begin{aligned}
&= (k\mathbf{u}) \cdot \mathbf{v} \\
&= u_x(kv_x) + u_y(kv_y) + u_z(kv_z) \\
&= \mathbf{u} \cdot (k\mathbf{v})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} \cdot \mathbf{v} &= v_x v_x + v_y v_y + v_z v_z \\
&= \sqrt{v_x^2 + v_y^2 + v_z^2}^{\,2} \\
&= \|\mathbf{v}\|^2
\end{aligned}
$$

$$\mathbf{0} \cdot \mathbf{v} = 0v_x + 0v_y + 0v_z = 0$$

10. Use the law of cosines ($c^2 = a^2 + b^2 - 2ab\cos\theta$, where a, b, and c are the lengths of the sides of a triangle and θ is the angle between sides a and b) to show

$$u_x v_x + u_y v_y + u_z v_z = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Hint: Consider Figure 1.9 and set $c^2 = \|\mathbf{u} - \mathbf{v}\|^2$, $a^2 = \|\mathbf{u}\|^2$ and $b^2 = \|\mathbf{v}\|^2$, and use the dot product properties from the previous exercise.

Solution:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\theta \\
\|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \\
(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \\
\mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta \\
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta
\end{aligned}
$$

11. Let $\mathbf{n} = (-2, 1)$. Decompose the vector $\mathbf{g} = (0, -9.8)$ into the sum of two orthogonal vectors, one parallel to $\mathbf{n}$ and the other orthogonal to $\mathbf{n}$. Also, draw the vectors relative to a 2D coordinate system.

Solution:

$$\mathbf{g}_{\parallel} = \mathrm{proj}_{\mathbf{n}}(\mathbf{g}) = \frac{(\mathbf{g} \cdot \mathbf{n})}{\|\mathbf{n}\|^2}\mathbf{n} = \frac{-9.8}{5}(-2,1) = -1.96(-2,1) = (3.92, -1.96)$$

$$\mathbf{g}_{\perp} = \mathbf{g} - \mathbf{g}_{\parallel} = (0, -9.8) - (3.92, -1.96) = (-3.92, -7.84)$$

12. Let $\mathbf{u} = (-2, 1, 4)$ and $\mathbf{v} = (3, -4, 1)$. Find $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, and show $\mathbf{w} \cdot \mathbf{u} = 0$ and $\mathbf{w} \cdot \mathbf{v} = 0$.

Solution:

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \left(u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x\right)$$

$$= (1 + 16, 12 + 2, 8 - 3)$$
$$= (17, 14, 5)$$

$$\mathbf{w} \cdot \mathbf{u} = 17(-2) + 14(1) + 5(4) = -34 + 14 + 20 = 0$$

$$\mathbf{w} \cdot \mathbf{v} = 17(3) + 14(-4) + 5(1) = 51 - 56 + 5 = 0$$

13. Let the following points define a triangle relative to some coordinate system: $\mathbf{A} = (0, 0, 0)$, $\mathbf{B} = (0, 1, 3)$, and $\mathbf{C} = (5, 1, 0)$. Find a vector orthogonal to this triangle. *Hint*: Find two vectors on two of the triangle's edges and use the cross product.

Solution:

$$\mathbf{u} = \mathbf{B} - \mathbf{A} = (0, 1, 3)$$
$$\mathbf{v} = \mathbf{C} - \mathbf{A} = (5,1,0)$$

$$\begin{aligned} \mathbf{n} = \mathbf{u} \times \mathbf{v} &= \left(u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x\right) \\ &= (0 - 3, 15 - 0, 0 - 5) \\ &= (-3, 15, -5) \end{aligned}$$

14. Prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$. *Hint*: Start with $\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta$ and use the trigonometric identity $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin\theta = \sqrt{1 - \cos^2\theta}$; then apply Equation 1.4.

Solution:

To make the derivation simpler, we compute the following three formulas up front:

$$\begin{aligned} \|\mathbf{u}\|^2\|\mathbf{v}\|^2 &= \left(u_x^2 + u_y^2 + u_z^2\right)\left(v_x^2 + v_y^2 + v_z^2\right) \\ &= u_x^2\left(v_x^2 + v_y^2 + v_z^2\right) + u_y^2\left(v_x^2 + v_y^2 + v_z^2\right) + u_z^2\left(v_x^2 + v_y^2 + v_z^2\right) \\ &= u_x^2 v_x^2 + u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_y^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 + u_z^2 v_z^2 \end{aligned} \quad (1)$$

$$\begin{aligned} (\mathbf{u} \cdot \mathbf{v})^2 &= \left(u_x v_x + u_y v_y + u_z v_z\right)\left(u_x v_x + u_y v_y + u_z v_z\right) \\ &= u_x v_x\left(u_x v_x + u_y v_y + u_z v_z\right) + u_y v_y\left(u_x v_x + u_y v_y + u_z v_z\right) \\ &\quad + u_z v_z\left(u_x v_x + u_y v_y + u_z v_z\right) \\ &= u_x v_x u_x v_x + u_x v_x u_y v_y + u_x v_x u_z v_z + u_y v_y u_x v_x + u_y v_y u_y v_y \\ &\quad + u_y v_y u_z v_z + u_z v_z u_x v_x + u_z v_z u_y v_y + u_z v_z u_z v_z \\ &= u_x^2 v_x^2 + 2u_x v_x u_y v_y + 2u_x v_x u_z v_z + u_y^2 v_y^2 + 2u_y v_y u_z v_z + u_z^2 v_z^2 \end{aligned} \quad (2)$$

$$\begin{aligned} \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = {} & u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 \\ & -2u_x v_x u_y v_y - 2u_x v_x u_z v_z - 2u_y v_y u_z v_z \end{aligned} \quad (3)$$

$$= \left(u_y^2 v_z^2 - 2u_y v_y u_z v_z + u_z^2 v_y^2\right) + (u_z^2 v_x^2 - 2u_x v_x u_z v_z + u_x^2 v_z^2) + \left(u_x^2 v_y^2 + -2u_x v_x u_y v_y + u_y^2 v_x^2\right)$$

Now,

$$\begin{aligned}\|\mathbf{u}\|\|\mathbf{v}\| \sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\cos^2\theta} \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1-\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\ &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{\frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\ &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}\end{aligned}$$

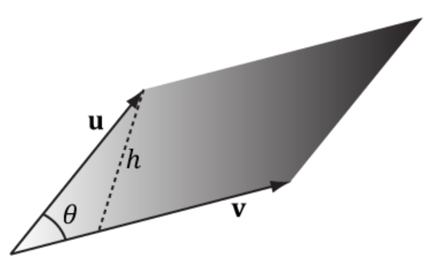
And

$$\|\mathbf{u}\times\mathbf{v}\| = \left\|u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x\right\|$$
$$= \sqrt{\left(u_y v_z - u_z v_y\right)^2 + (u_z v_x - u_x v_z)^2 + \left(u_x v_y - u_y v_x\right)^2}$$
$$= \sqrt{\left(u_y^2 v_z^2 - 2u_y v_z u_z v_y + u_z^2 v_y^2\right) + (u_z^2 v_x^2 - 2u_z v_x u_x v_z + u_x^2 v_z^2) + \left(u_x^2 v_y^2 - 2u_x v_y u_y v_x + u_y^2 v_x^2\right)}$$
$$= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2}$$

Thus we obtain the desired result:

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

15. Prove that $\|\mathbf{u}\times\mathbf{v}\|$ gives the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$; see Figure below.



Solution:

The area is the base times the height:

$$A = \|\mathbf{v}\|h$$

Using trigonometry, the height is given by $h = \|\mathbf{u}\|\sin(\theta)$. This, along with the application of Exercise 14, we can conclude:

$$A = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta) = \|\mathbf{u} \times \mathbf{v}\|$$

16. Give an example of 3D vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. This shows the cross product is generally not associative. *Hint*: Consider combinations of the simple vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$.

Solution:

Choose $\mathbf{u} = (1,1,0)$, $\mathbf{v} = \mathbf{i}$, and $\mathbf{w} = \mathbf{j}$. Then:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (1,1,0) \times \mathbf{k} = (1, -1,0)$$

But,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$$

17. Prove that the cross product of two nonzero parallel vectors results in the null vector; that is, $\mathbf{u} \times k\mathbf{u} = 0$. *Hint*: Just use the cross product definition.

Solution:

$$\begin{aligned} \mathbf{u} \times k\mathbf{u} &= \left(u_y k u_z - u_z k u_y, u_z k u_x - u_x k u_z, u_x k u_y - u_y k u_x\right) \\ &= \left(k u_y u_z - k u_z u_y, k u_z u_x - k u_x u_z, k u_x u_y - k u_y u_x\right) \\ &= \mathbf{0} \end{aligned}$$