Introduction to 3D Game Programming with DirectX 9.0c: A Shader Approach

Part I Solutions

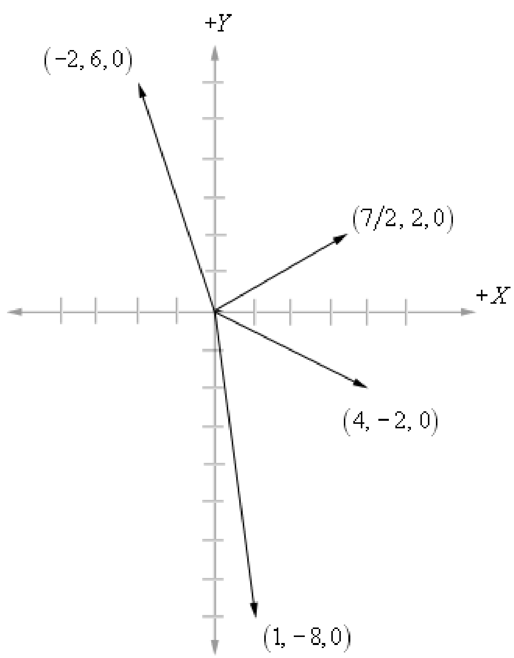
Note 1: Please email to frank@moon-labs.com if you find any errors.

Note 2: Use only after you have tried, and struggled with, the problems yourself.

Chapter 1 Vector Algebra

1. Let $\vec{u} = (1, 2, 0)$ and $\vec{v} = (3, -4, 0)$. Compute $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $2\vec{u} + 1/2\,\vec{v}$, and $-2\vec{u} + \vec{v}$ and draw the vectors relative to a coordinate system.

$$\vec{u} + \vec{v} = (1, 2, 0) + (3, -4, 0) = (1+3, 2+(-4), 0+0) = (4, -2, 0)$$
$$\vec{u} - \vec{v} = (1, 2, 0) - (3, -4, 0) = (1-3, 2-(-4), 0-0) = (-2, 6, 0)$$
$$2\vec{u} + 1/2\,\vec{v} = 2(1, 2, 0) + 1/2(3, -4, 0) = (2+3/2, 4+(-2), 0+0) = (7/2, 2, 0)$$
$$-2\vec{u} + \vec{v} = -2(1, 2, 0) + (3, -4, 0) = (-2+3, -4+(-4), 0+0) = (1, -8, 0)$$



2. Let $\vec{u} = (-2, 1, 4)$ and $\vec{v} = (3, -4, 1)$. Normalize $\vec{u}$ and $\vec{v}$.

$$\vec{u}_n = \frac{\vec{u}}{\|\vec{u}\|} = \frac{(-2, 1, 4)}{\sqrt{(-2)^2 + 1^2 + 4^2}} = \frac{(-2, 1, 4)}{\sqrt{21}} = \left(\frac{-2}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$$

$$\vec{v}_n = \frac{\vec{v}}{\|\vec{v}\|} = \frac{(3, -4, 1)}{\sqrt{3^2 + (-4)^2 + 1^2}} = \frac{(3, -4, 1)}{\sqrt{26}} = \left(\frac{3}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$$

3. Show $\vec{u}/\|\vec{u}\|$ has a length of one unit. (Hint: Compute the length of $\vec{u}/\|\vec{u}\|$.)

The length of $\vec{u}/\|\vec{u}\| = \left(u_x/\|\vec{u}\|, u_y/\|\vec{u}\|, u_z/\|\vec{u}\|\right)$ is given by:

$$\sqrt{\left(\frac{u_x}{\|\vec{u}\|}\right)^2 + \left(\frac{u_y}{\|\vec{u}\|}\right)^2 + \left(\frac{u_z}{\|\vec{u}\|}\right)^2} = \sqrt{\frac{u_x^2 + u_y^2 + u_z^2}{\|\vec{u}\|^2}} = \frac{\sqrt{u_x^2 + u_y^2 + u_z^2}}{\|\vec{u}\|} = \frac{\|\vec{u}\|}{\|\vec{u}\|} = 1.$$

4. Is the angle between $\vec{u}$ and $\vec{v}$ orthogonal, acute, or obtuse?

a. $\vec{u} = (1, 1, 1)$, $\vec{v} = (2, 2, 2)$

b. $\vec{u} = (1, 1, 0)$, $\vec{v} = (-2, 2, 0)$

c. $\vec{u} = (-1, -1, -1)$, $\vec{v} = (3, 1, 0)$

Use the geometric properties of the dot product given on page 10:

$$(1, 1, 1) \cdot (2, 2, 2) = 1 \cdot 2 + 1 \cdot 2 + 1 \cdot 2 = 6 > 0 \Rightarrow \text{ acute angle}$$
$$(1, 1, 0) \cdot (-2, 2, 0) = 1 \cdot (-2) + 1 \cdot 2 + 0 \cdot 0 = 0 \Rightarrow \text{ orthogonal}$$
$$(-1, -1, -1) \cdot (3, 1, 0) = (-1) \cdot 3 + (-1) \cdot 1 + (-1) \cdot 0 = -4 < 0 \Rightarrow \text{ obtuse angle}$$

5. Let $\vec{u} = (-2, 1, 4)$ and $\vec{v} = (3, -4, 1)$. Find the angle θ between $\vec{u}$ and $\vec{v}$.

Use Equation 1.4:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\theta = \cos^{-1}\frac{(-2,1,4)\cdot(3,-4,1)}{\sqrt{(-2)^2+1^2+4^2}\sqrt{3^2+(-4)^2+1^2}}$$

$$\theta = \cos^{-1}\frac{-6-4+4}{\sqrt{21}\sqrt{26}}$$

$$\theta = \cos^{-1}\frac{-6}{\sqrt{546}}$$

$$\theta \approx 104.88°$$

6. Let $\vec{u} = (u_x, u_y, u_z)$, $\vec{v} = (v_x, v_y, v_z)$, and $\vec{w} = (w_x, w_y, w_z)$, show that the following properties are true for $\mathbb{R}^3$:

a. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

b. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

c. $k(\vec{u}\cdot\vec{v}) = (k\vec{u})\cdot\vec{v} = \vec{u}\cdot(k\vec{v})$

d. $\vec{v}\cdot\vec{v} = \|\vec{v}\|^2$

e. $\vec{0}\cdot\vec{v} = 0$

(Hint: Just use the definition, for example,
$\vec{v}\cdot\vec{v} = v_x v_x + v_y v_y + v_z v_z = v_x^2 + v_y^2 + v_z^2 = \left(\sqrt{v_x^2+v_y^2+v_z^2}\right)^2 = \left(\|\vec{v}\|\right)^2$.)

a)

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (u_x, u_y, u_z)\cdot(v_x, v_y, v_z)\\ &= u_x v_x + u_y v_y + u_z v_z\\ &= v_x u_x + v_y u_y + v_z u_z\\ &= (v_x, v_y, v_z)\cdot(u_x, u_y, u_z)\\ &= \vec{v}\cdot\vec{u}\end{aligned}$$

b)

$$\begin{aligned}\vec{u}\cdot(\vec{v}+\vec{w}) &= (u_x, u_y, u_z)\cdot\left[(v_x, v_y, v_z)+(w_x, w_y, w_z)\right]\\ &= (u_x, u_y, u_z)\cdot(v_x+w_x, v_y+w_y, v_z+w_z)\\ &= u_x(v_x+w_x)+u_y(v_y+w_y)+u_z(v_z+w_z)\\ &= u_xv_x+u_xw_x+u_yv_y+u_yw_y+u_zv_z+u_zw_z\\ &= (u_xv_x+u_yv_y+u_zv_z)+(u_xw_x+u_yw_y+u_zw_z)\\ &= \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}\end{aligned}$$

c)

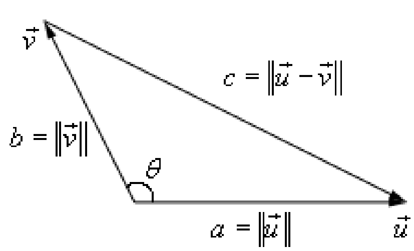
$$\begin{aligned}k(\vec{u}\cdot\vec{v}) &= k(u_xv_x+u_yv_y+u_zv_z)\\ &= ku_xv_x+ku_yv_y+ku_zv_z\\ &= (ku_x)v_x+(ku_y)v_y+(ku_z)v_z\\ &= (k\vec{u})\cdot\vec{v}\end{aligned}$$

$$\begin{aligned}k(\vec{u}\cdot\vec{v}) &= k(u_xv_x+u_yv_y+u_zv_z)\\ &= ku_xv_x+ku_yv_y+ku_zv_z\\ &= u_x(kv_x)+u_y(kv_y)+u_z(kv_z)\\ &= \vec{u}\cdot(k\vec{v})\end{aligned}$$

d) $\vec{v}\cdot\vec{v} = v_xv_x+v_yv_y+v_zv_z = v_x^2+v_y^2+v_z^2 = \left(\sqrt{v_x^2+v_y^2+v_z^2}\right)^2 = (\|\vec{v}\|)^2$

e) $\vec{0}\cdot\vec{v} = (0, 0, 0)\cdot(v_x, v_y, v_z) = 0v_x+0v_y+0v_z = 0$

7. Use the Law of Cosines ($c^2 = a^2+b^2-2ab\cos\theta$, where *a*, *b*, and *c* are the lengths of the sides of a triangle and θ is the angle between sides *a* and *b*) to show $u_xv_x+u_yv_y+u_zv_z = \|\vec{u}\|\|\vec{v}\|\cos\theta$. (Hint: Draw a picture and set $c^2 = \|\vec{u}-\vec{v}\|^2$, $a^2 = \|\vec{u}\|^2$, and $b^2 = \|\vec{v}\|^2$, and use the dot product properties from the previous exercise.)



The figure shows the setup, then by the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\vec{u}\cdot\vec{u} - 2(\vec{u}\cdot\vec{v}) + \vec{v}\cdot\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$\|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$-2(\vec{u}\cdot\vec{v}) = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$(\vec{u}\cdot\vec{v}) = \|\vec{u}\|\|\vec{v}\|\cos\theta$$
$$u_x v_x + u_y v_y + u_z v_z = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

8. Let $\vec{v} = (4, 3, 0)$ and $\vec{n} = \left(2/\sqrt{5}, 1/\sqrt{5}, 0\right)$. Show that $\vec{n}$ is a unit vector and find the orthogonal projection, $\vec{p}$, of $\vec{v}$ on $\vec{n}$. Then find a vector $\vec{w}$ orthogonal to $\vec{n}$ such that $\vec{v} = \vec{p} + \vec{w}$. (Hint: Draw the vectors for insight, what does $\vec{v} - \vec{p}$ look like?)

We have,

$$\|\vec{n}\| = \sqrt{\left(2/\sqrt{5}\right)^2 + \left(1/\sqrt{5}\right)^2 + (0)^2} = \sqrt{\tfrac{4}{5} + \tfrac{1}{5}} = \sqrt{1} = 1.$$

So $\vec{n}$ is a unit vector. To find the projection, we use the equation below Figure 1.8:

$$\begin{aligned}\vec{p} &= (\vec{v}\cdot\vec{n})\vec{n} \\ &= \left[(4, 3, 0)\cdot\left(2/\sqrt{5}, 1/\sqrt{5}, 0\right)\right]\vec{n} \\ &= \left(\tfrac{8}{\sqrt{5}} + \tfrac{3}{\sqrt{5}}\right)\vec{n} \\ &= \tfrac{11}{\sqrt{5}}\vec{n} \\ &= \tfrac{11}{\sqrt{5}}\left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0\right) \\ &= \left(\tfrac{22}{5}, \tfrac{11}{5}, 0\right)\end{aligned}$$

If $\vec{v} = \vec{p} + \vec{w}$, then $\vec{w} = \vec{v} - \vec{p} = (4, 3, 0) - \left(\tfrac{22}{5}, \tfrac{11}{5}, 0\right) = \left(\tfrac{20}{5}, \tfrac{15}{5}, 0\right) - \left(\tfrac{22}{5}, \tfrac{11}{5}, 0\right) = \left(\tfrac{-2}{5}, \tfrac{4}{5}, 0\right)$. Moreover, because $\left(\tfrac{-2}{5}, \tfrac{4}{5}, 0\right)\cdot\left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0\right) = -\tfrac{2}{5}\tfrac{2}{\sqrt{5}} + \tfrac{4}{5}\tfrac{1}{\sqrt{5}} = 0$, we have that $\vec{w}$ is orthogonal to $\vec{n}$.

9. Let $\vec{u} = (-2, 1, 4)$ and $\vec{v} = (3, -4, 1)$. Find $\vec{w} = \vec{u}\times\vec{v}$, and show $\vec{w}\cdot\vec{u} = 0$ and $\vec{w}\cdot\vec{v} = 0$.

Apply Equation 1.5:

$$\begin{aligned}\vec{u}\times\vec{v} &= (-2, 1, 4)\times(3, -4, 1)\\ &= (1\cdot 1 - 4\cdot(-4), 4\cdot 3 - (-2)\cdot 1, -2(-4) - 1\cdot 3)\\ &= (17, 14, 5)\end{aligned}$$

$$\vec{w}\cdot\vec{u} = (17, 14, 5)\cdot(-2, 1, 4) = -2\cdot 17 + 14\cdot 1 + 5\cdot 4 = 0$$
$$\vec{w}\cdot\vec{v} = (17, 14, 5)\cdot(3, -4, 1) = 17\cdot 3 - 4\cdot 14 + 5\cdot 1 = 0$$

10. Let the following points define a triangle relative to some coordinate system: $\vec{A} = (0, 0, 0)$, $\vec{B} = (0, 1, 3)$, and $\vec{C} = (5, 1, 0)$. Find a vector orthogonal to this triangle. (Hint: Find two vectors on two of the triangle's edges and use the cross product.)

The two vectors on the edges of the triangle are:

$$\vec{u} = \vec{B} - \vec{A} = (0, 1, 3)$$
$$\vec{v} = \vec{C} - \vec{A} = (5, 1, 0)$$

Then a vector orthogonal to this triangle is given by:

$$\vec{u}\times\vec{v} = (0, 1, 3)\times(5, 1, 0) = (-3, 15, -5).$$

11. Suppose that we have frames A and B. Let $\vec{p}_A = (1, -2, 0)$ and $\vec{q}_A = (1, 2, 0)$ represent a point and force, respectively, relative to frame A. Moreover, let $\vec{O} = (-6, 2, 0)$, $\vec{u} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, $\vec{v} = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$, and $\vec{w} = (0, 0, 1)$ describe frame A relative to frame B. Find $\vec{p}_B = (x, y, z)$ and $\vec{q}_B = (x, y, z)$ that describe the point and force relative to frame B.

$$\vec{p}_B = 1\vec{u} - 2\vec{v} + 0\vec{w} + \vec{O} = \left(\tfrac{1}{\sqrt{2}} + \tfrac{2}{\sqrt{2}} - 6, \tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} + 2, 0\right) = \left(\tfrac{3-6\sqrt{2}}{\sqrt{2}}, \tfrac{2\sqrt{2}-1}{\sqrt{2}}, 0\right)$$
$$\vec{q}_B = 1\vec{u} + 2\vec{v} + 0\vec{w} = \left(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}}, \tfrac{1}{\sqrt{2}} + \tfrac{2}{\sqrt{2}}, 0\right) = \left(\tfrac{-1}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}, 0\right)$$

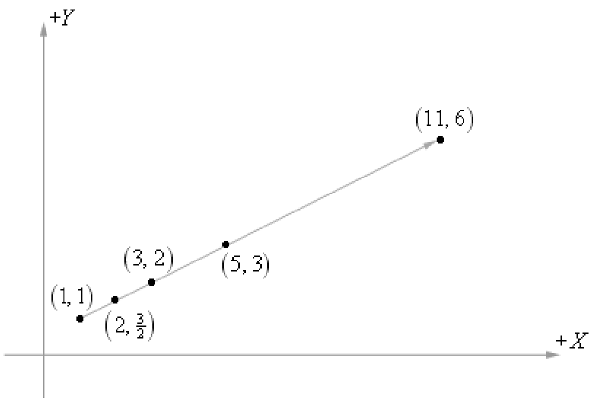
12. Let $\vec{p}(t) = (1, 1) + t(2, 1)$ be a ray relative to some coordinate system. Plot the points on the ray at t = 0.0, 0.5, 1.0, 2.0, and 5.0.

The points are:

$$\vec{p}(t) = (1, 1) + 0(2, 1) = (1, 1)$$
$$\vec{p}(t) = (1, 1) + \tfrac{1}{2}(2, 1) = (2, 3/2)$$

$$\vec{p}(t) = (1,1) + 1(2,1) = (3,2)$$
$$\vec{p}(t) = (1,1) + 2(2,1) = (5,3)$$
$$\vec{p}(t) = (1,1) + 5(2,1) = (11,6)$$

And the plot:



13. Let $\vec{p}_0$ and $\vec{p}_1$ define the endpoints of a line segment. Show that the equation for a line segment can also be written as $\vec{p}(t) = (1-t)\vec{p}_0 + t\vec{p}_1$ for $t \in [0,1]$.

$$\vec{p}(t) = \vec{p}_0 + t(\vec{p}_1 - \vec{p}_0) = \vec{p}_0 + t\vec{p}_1 - t\vec{p}_0 = (1-t)\vec{p}_0 + t\vec{p}_1$$

14. Rewrite the program in §1.8 twice; first using 2D vectors (`D3DXVECTOR2`) and second using 4D vectors (`D3DXVECTOR4`). (Hint: Search the index for these keywords in the DirectX SDK documentation: `D3DXVECTOR2`, `D3DXVECTOR4`, `D3DXVec2`, and `D3DXVec4`.)

We rewrite the program using `D3DXVECTOR2`; the other case is analogous.

```
#include <d3dx9.h>
#include <iostream>
using namespace std;

// Overload the  "<<" operators so that we can use cout to
// output D3DXVECTOR2 objects.

ostream& operator<<(ostream& os, D3DXVECTOR2& v)
{
      os << "(" << v.x << ", " << v.y << ")";
      return os;
}

int main()
{
      // Using constructor, D3DXVECTOR2(FLOAT x, FLOAT y);
      D3DXVECTOR2 u(1.0f, 2.0f);
```

```
    // Using constructor, D3DXVECTOR2(CONST FLOAT *);
    float x[2] = {-2.0f, 1.0f};
    D3DXVECTOR2 v(x);

    // Using default constructor, D3DXVECTOR2();
    D3DXVECTOR2 a, b, c, d;

    // Vector addition: D3DXVECTOR2 operator +
    a = u + v;

    // Vector subtraction: D3DXVECTOR2 operator -
    b = u - v;

    // Scalar multiplication: D3DXVECTOR2 operator *
    c = u * 10;

    // ||u||
    float length = D3DXVec2Length(&u);

    // d = u / ||u||
    D3DXVec2Normalize(&d, &u);

    // s = u dot v
    float s = D3DXVec2Dot(&u, &v);

    cout << "u       = " << u << endl;
    cout << "v       = " << v << endl;
    cout << "a       = " << a << endl;
    cout << "b       = " << b << endl;
    cout << "c       = " << c << endl;
    cout << "d       = " << d << endl;
    cout << "||u||   = " << length << endl;
    cout << "u dot v = " << s << endl;

    return 0;
}
```

Chapter 2 Matrix Algebra

1. Let

$$S = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix}, \text{ and } \vec{u} = \begin{bmatrix} 2 & -1 & 1 & 1 \end{bmatrix}.$$

Compute the following matrix products: ST, TS, $\vec{u}S$, $\vec{u}T$, and $\vec{u}(ST)$. Does $ST = TS$?

Just use the definition of matrix multiplication (i.e., Equation 2.1):

$$ST = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 6 & 10 & -4 & 1 \end{bmatrix}$$

$$\vec{u}S = \begin{bmatrix} 2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 & 1 \end{bmatrix}$$

$$\vec{u}T = \begin{bmatrix} 2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 0 & 1 \end{bmatrix}$$

$$\vec{u}(ST) = \begin{bmatrix} 2 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 3 & 1 \end{bmatrix}$$

$ST \neq TS$.

2. Show $\vec{u}\vec{v} = \begin{bmatrix} u_x, & u_y, & u_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \vec{u} \cdot \vec{v}$.

The notation $\vec{u}\vec{v}$ means the product of a 1×3 row vector $\vec{u}$ with a 3×1 column vector $\vec{v}$ so that the matrix product is defined. By the definition of matrix multiplication we have that the product of a 1×3 matrix with a 3×1 matrix is a 1×1 matrix; we can think of a 1×1 matrix as just a scalar. Applying Equation 2.1 we obtain:

$$\begin{bmatrix} u_x, & u_y, & u_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} u_x v_x + u_y v_y + u_z v_z \end{bmatrix} = u_x v_x + u_y v_y + u_z v_z = \vec{u} \cdot \vec{v}$$

This exercise simply shows that we can express a dot product in matrix notation using matrix multiplication.

3. Using S and T from Exercise 1, compute S^T, T^T, and $(ST)^T$. What is $(S^T)^T$ and $(T^T)^T$? Does $(ST)^T = T^T S^T$?

$$S^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = S$$

Observe for this special matrix, $S^T = S$. When a matrix equals its transpose, the matrix is said to by *symmetric*.

$$T^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(ST)^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & -2 & 0 & -5 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(S^T)^T = \left(\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \right)^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = S$$

$$(T^T)^T = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix}^T \right)^T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -5 & -1 & 1 \end{bmatrix} = T$$

So if we take one transpose, we interchange the rows and columns. If we take another transpose, we interchange the rows and columns again and end up back to where we started.

$$T^T S^T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & -2 & 0 & -5 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (ST)^T .$$

4. Using S and T from Exercise 1, verify that $(ST)^{-1} = T^{-1}S^{-1}$. (Use `D3DXMatrixInverse` to do the calculations.)

Using `D3DXMatrixInverse`, we find that:

$$(ST)^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{-2}{3} & \frac{-5}{2} & \frac{1}{4} & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 5 & 1 & 1 \end{bmatrix}$$

Using the definition of matrix multiplication, we compute

$$T^{-1}S^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{-2}{3} & \frac{-5}{2} & \frac{1}{4} & 1 \end{bmatrix}.$$

We see that indeed $(ST)^{-1} = T^{-1}S^{-1}$.

5. Write the following linear combination as a vector-matrix multiplication: $\vec{v} = 2(1, 2, 3) + -4(-5, 0, -1) + 3(2, -2, -3)$.

Observe:

$$\begin{aligned} \vec{v} &= 2(1, 2, 3) + -4(-5, 0, -1) + 3(2, -2, -3) \\ &= (2, 4, 6) + (20, 0, 4) + (6, -6, -9) \\ &= (28, -2, 1) \end{aligned}$$

By Equation 2.2, we can write this linear combination as a vector-matrix product:

$$\vec{v} = [2, -4, 3] \begin{bmatrix} 1 & 2 & 3 \\ -5 & 0 & -1 \\ 2 & -2 & -3 \end{bmatrix} = [28, -2, 1]$$

6. Redo Exercise 11 from Chapter 1 using Equation 2.3.

The idea here is to just express the change of frame calculation by a matrix equation, namely Equation 2.3. To change from frame *A* to frame *B*, we stick the frame *A* vectors, with coordinates relative to frame *B* in homogeneous coordinates, $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{O}$ into the rows of a matrix *C*. Then, given the vector/point $\vec{p}_A = (x, y, z, w)$ that specifies a vector/point relative to a frame *A*, we obtain the same vector/point, identified by $\vec{p}_B = (x, y, z, w)$ relative to frame *B*, by performing the vector-matrix multiplication: $\vec{p}_B = \vec{p}_A C$.

From Exercise 11 of Chapter 1, we have, in homogenous coordinates, the frame *A* vectors (relative to frame *B*):

$$\vec{u} = \left(1/\sqrt{2}, 1/\sqrt{2}, 0, 0\right),\ \vec{v} = \left(-1/\sqrt{2}, 1/\sqrt{2}, 0, 0\right),\ \vec{w} = (0, 0, 1, 0),\ \vec{O} = (-6, 2, 0, 1)$$

And, again in homogeneous coordinates, $\vec{p}_A = (1, -2, 0, 1)$ and $\vec{q}_A = (1, 2, 0, 0)$.

Then the change of frame transformation may be computed as follows:

$$\vec{p}_B = [1, -2, 0, 1]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{bmatrix} = \left(\tfrac{3-6\sqrt{2}}{\sqrt{2}}, \tfrac{2\sqrt{2}-1}{\sqrt{2}}, 0, 1\right)$$

$$\vec{q}_B = [1, 2, 0, 0]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{bmatrix} = \left(\tfrac{-1}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}, 0, 0\right)$$

7. Show that

$$AB = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} = \begin{bmatrix} \vec{u}_{row1} \\ \vec{u}_{row2} \\ \vec{u}_{row3} \end{bmatrix}\begin{bmatrix} \vec{v}_{col1} & \vec{v}_{col2} & \vec{v}_{col3} \end{bmatrix} = \begin{bmatrix} \vec{u}_{row1}B \\ \vec{u}_{row2}B \\ \vec{u}_{row3}B \end{bmatrix}.$$

This result shows that a matrix-matrix multiplication can be viewed as several linear combinations; specifically, in this case, the matrix product AB is essentially the three linear combinations $\vec{u}_{row1}B$, $\vec{u}_{row2}B$, and $\vec{u}_{row3}B$.

One way to do this is to just do the matrix multiplication brute force, but it is tedious even for small 3×3 matrices. To make things simpler, we will only calculate the *i*th row of AB, where i is arbitrarily 1, 2, or 3.

$$
\begin{aligned}
(AB)_{row\,i} &= \begin{bmatrix} \vec{u}_{row\,i} \cdot \vec{v}_{col1} & \vec{u}_{row\,i} \cdot \vec{v}_{col2} & \vec{u}_{row\,i} \cdot \vec{v}_{col3} \end{bmatrix} \\
&= \begin{bmatrix} u_{i1} & u_{i2} & u_{i3} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \\
&= \vec{u}_{row\,i} B
\end{aligned}
$$

Now we just let i vary over 1, 2, and 3 to obtain the result:

$$
AB = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} = \begin{bmatrix} \vec{u}_{row1} B \\ \vec{u}_{row2} B \\ \vec{u}_{row3} B \end{bmatrix}
$$

Chapter 3 Transformations; Planes

1. Show that the *x*-axis rotation transformation given by Equation 3.7 is a linear transformation.

Recall $R_x(\vec{u}) = (x,\ y\cos\theta - z\sin\theta,\ y\sin\theta + z\cos\theta)$. We must show that this transformation satisfies the linearity condition, which is Equation 3.1. (To save horizontal space, we will use column vector notation.)

$$R_x(\alpha\vec{u}+\beta\vec{v}) = \begin{bmatrix} \alpha u_x + \beta v_x \\ (\alpha u_y + \beta v_y)\cos\theta - (\alpha u_z + \beta v_z)\sin\theta \\ (\alpha u_y + \beta v_y)\sin\theta + (\alpha u_z + \beta v_z)\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha u_x + \beta v_x \\ \alpha u_y \cos\theta + \beta v_y \cos\theta - \alpha u_z \sin\theta - \beta v_z \sin\theta \\ \alpha u_y \sin\theta + \beta v_y \sin\theta + \alpha u_z \cos\theta + \beta v_z \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha u_x \\ \alpha u_y \cos\theta - \alpha u_z \sin\theta \\ \alpha u_y \sin\theta + \alpha u_z \cos\theta \end{bmatrix} + \begin{bmatrix} \beta v_x \\ \beta v_y \cos\theta - \beta v_z \sin\theta \\ \beta v_y \sin\theta + \beta v_z \cos\theta \end{bmatrix}$$

$$= \alpha \begin{bmatrix} u_x \\ u_y \cos\theta - u_z \sin\theta \\ u_y \sin\theta + u_z \cos\theta \end{bmatrix} + \beta \begin{bmatrix} v_x \\ v_y \cos\theta - v_z \sin\theta \\ v_y \sin\theta + v_z \cos\theta \end{bmatrix}$$

$$= \alpha R_x(\vec{u}) + \beta R_x(\vec{v})$$

We have shown $R_x(\vec{u})$ to be a linear transformation.

2. Show that the identity function, defined by $I(\vec{u}) = \vec{u}$, is a linear transformation, and show that its matrix representation is the identity matrix.

We have $I(\alpha\vec{u}+\beta\vec{v}) = \alpha\vec{u}+\beta\vec{v} = \alpha \cdot I(\vec{u}) + \beta \cdot I(\vec{v})$, so the definition 3.1 of a linear transformation is satisfied. By applying the linear transformation to each of the basis vectors and then putting them into the rows of a matrix, we obtain the matrix representation of the identity function:

$$\begin{bmatrix} I(\vec{i}) \\ I(\vec{j}) \\ I(\vec{k}) \end{bmatrix} = \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Show that the row vectors in the y-axis rotation matrix R_y are orthonormal.

Recall $R_y = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$.

To show the row vectors are orthonormal, we must show they all are of unit length and that they are all mutually orthogonal. Let $\vec{r}_1 = (\cos\theta, 0, -\sin\theta)$, $\vec{r}_2 = (0, 1, 0)$, and $\vec{r}_3 = (\sin\theta, 0, \cos\theta)$ be the row vectors of the matrix R_y. By the well known trig identity $\cos^2\theta + \sin^2\theta = 1$, we have:

$$\|\vec{r}_1\| = \sqrt{\cos^2\theta + 0^2 + (-\sin^2\theta)} = 1$$
$$\|\vec{r}_2\| = \sqrt{0^2 + 1^2 + 0^2} = 1$$
$$\|\vec{r}_3\| = \sqrt{\sin^2\theta + 0^2 + \cos^2\theta} = 1$$

So all the row vectors are of unit length. Now a straightforward application of the dot product shows:

$$\vec{r}_1 \cdot \vec{r}_2 = 0$$
$$\vec{r}_1 \cdot \vec{r}_3 = 0$$
$$\vec{r}_2 \cdot \vec{r}_3 = 0$$

Thus the row vectors are all orthogonal to each other. We have shown the row vectors to be orthonormal.

4. Let R_y be the y-axis rotation matrix. Show that the transpose of this matrix is its inverse; that is, show $R_y R_y^T = R_y^T R_y = I$.

$$R_y^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_y R_y^T = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$R_y^T R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

The inverse is unique so we must have $R_y^T = R_y^{-1}$.

5. In §3.1.4, we showed how the x-axis, y-axis, and z-axis rotation matrices could be derived directly. Another perspective is to think of these rotation matrices as special

cases of the arbitrary axis rotation matrix. Show that the arbitrary axis rotation matrix reduces to the x-axis, y-axis, and z-axis rotation matrices when $\vec{q}$ equals, $\vec{i}$, $\vec{j}$, and $\vec{k}$ (i.e., the standard basis vectors), respectively.

We have $R_{\vec{q}} = \begin{bmatrix} c + x^2(1-c) & xy(1-c) + zs & xz(1-c) - ys \\ xy(1-c) - zs & c + y^2(1-c) & yz(1-c) + xs \\ xz(1-c) + ys & yz(1-c) - xs & c + z^2(1-c) \end{bmatrix}$.

Taking $\vec{q} = \vec{i} = (1, 0, 0)$, we have in the above matrix $x = 1,\ y = z = 0$, and matrix reduces to:

$$R_{\vec{i}} = \begin{bmatrix} c + 1(1-c) & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = R_x.$$

The process is analogous to obtain the y-axis, and z-axis rotation matrices.

6. Show that the translation matrix affects points, but not vectors.

Using the definition of matrix multiplication (Equation 2.1), we have:

$$[x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix}$$
$$= \left[(x,y,z,1)\cdot(1,0,0,b_x),(x,y,z,1)\cdot(0,1,0,b_y),(x,y,z,1)\cdot(0,0,1,b_z),(x,y,z,1)\cdot(0,0,0,1)\right]$$
$$= \left[x + b_x,\ y + b_y,\ z + b_z,\ 1\right]$$

and

$$[x, y, z, 0]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix}$$
$$= \left[(x,y,z,0)\cdot(1,0,0,b_x),(x,y,z,0)\cdot(0,1,0,b_y),(x,y,z,0)\cdot(0,0,1,b_z),(x,y,z,0)\cdot(0,0,0,1)\right]$$
$$= [x, y, z, 0]$$

7. Verify that the given scaling matrix inverse is indeed the inverse of the scaling matrix; that is, show, by directly doing the matrix multiplication, that $SS^{-1} = S^{-1}S = I$. Similarly, verify that the given translation matrix inverse is indeed the inverse of the translation matrix; that is, show, by directly doing the matrix multiplication, that $TT^{-1} = T^{-1}T = I$.

$$SS^{-1} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s_x}{s_x} & 0 & 0 & 0 \\ 0 & \frac{s_y}{s_y} & 0 & 0 \\ 0 & 0 & \frac{s_z}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$S^{-1}S = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{s_x}{s_x} & 0 & 0 & 0 \\ 0 & \frac{s_y}{s_y} & 0 & 0 \\ 0 & 0 & \frac{s_z}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$TT^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -b_x & -b_y & -b_z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x - b_x & b_y - b_y & b_z - b_z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$T^{-1}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -b_x & -b_y & -b_z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x & b_y & b_z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b_x - b_x & b_y - b_y & b_z - b_z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

8. Let $\vec{p}_0 = (0, 1, 0)$, $\vec{p}_1 = (-1, 3, 6)$, and $\vec{p}_2 = (8, 5, 3)$ be three points. Find the plane these points define.

Two vectors on the plane are given by:

$$\vec{u} = \vec{p}_1 - \vec{p}_0 = (-1, 3, 6) - (0, 1, 0) = (-1, 2, 6)$$
$$\vec{v} = \vec{p}_2 - \vec{p}_0 = (8, 5, 3) - (0, 1, 0) = (8, 4, 3)$$

Now take the cross product to get a vector perpendicular to the plane (i.e., the plane normal):

$$\vec{n} = \vec{u} \times \vec{v} = (-1, 2, 6) \times (8, 4, 3) = (-18, 51, -20).$$

Moreover, $d = -(\vec{n} \cdot \vec{p}_0) = -51$. Then the plane consists of all the points $\vec{p} = (x, y, z)$ that satisfy the equation:

$$\vec{n} \cdot (\vec{p} - \vec{p}_0) = \vec{n} \cdot \vec{p} - (\vec{n} \cdot \vec{p}_0) = -18x + 51y - 20z - 51 = 0.$$

Remark: Because we didn't normalize the plane normal, the value d is no longer the signed distance from the origin, but some scaled distance.

9. Let $\pi = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -5\right)$ be a plane. Define the locality of the following points relative to the plane: $\left(3\sqrt{3}, 5\sqrt{3}, 0\right)$, $\left(2\sqrt{3}, \sqrt{3}, 2\sqrt{3}\right)$, and $\left(\sqrt{3}, -\sqrt{3}, 0\right)$.

The plane equation is: $\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z - 5 = 0$. Plugging these points into the left-hand side of the equation gives:

$$\frac{1}{\sqrt{3}}\left(3\sqrt{3}\right) + \frac{1}{\sqrt{3}}\left(5\sqrt{3}\right) + \frac{1}{\sqrt{3}}(0) - 5 = 3 \Rightarrow \text{In front of the plane}$$
$$\frac{1}{\sqrt{3}}\left(2\sqrt{3}\right) + \frac{1}{\sqrt{3}}\left(\sqrt{3}\right) + \frac{1}{\sqrt{3}}\left(2\sqrt{3}\right) - 5 = 0 \Rightarrow \text{On the plane}$$
$$\frac{1}{\sqrt{3}}\left(\sqrt{3}\right) + \frac{1}{\sqrt{3}}\left(-\sqrt{3}\right) + \frac{1}{\sqrt{3}}(0) - 5 = -5 \Rightarrow \text{In back of the plane}$$

10. Let $\pi = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -5\right)$ be a plane, and let $\vec{r}(t) = (-1, 1, -1) + t(1, 0, 0)$ be a ray. Find the point at which the ray intersects the plane. Then write a short program using the `D3DXPlaneIntersectLine` (see the SDK documentation for the prototype) function to verify your answer.

$$t = \frac{-d - (\vec{n} \cdot \vec{p}_0)}{(\vec{n} \cdot \vec{u})} = \frac{5 - \left(\frac{-1}{\sqrt{3}}\right)}{\frac{1}{\sqrt{3}}} = 5\sqrt{3} + 1$$

$$\begin{aligned} \vec{r}\left(5\sqrt{3} + 1\right) &= (-1, 1, -1) + \left(5\sqrt{3} + 1\right)(1, 0, 0) \\ &= (-1, 1, -1) + \left(5\sqrt{3} + 1, 0, 0\right) \\ &= \left(5\sqrt{3}, 1, -1\right) \end{aligned}$$

We plug $\left(5\sqrt{3}, 1, -1\right)$ into the plane equation to verify it indeed lies on the plane:

$$\frac{1}{\sqrt{3}}5\sqrt{3}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}-5=0 \Rightarrow \text{On the plane}$$

The code is given by:

```
#include <d3dx9.h>
#include <iostream>
using namespace std;

// Overload the  "<<" operators so that we can use cout to
// output D3DXVECTOR3 objects.

ostream& operator<<(ostream& os, D3DXVECTOR3& v)
{
	os << "(" << v.x << ", " << v.y << ", " << v.z << ")";
	return os;
}

int main()
{
	D3DXVECTOR3 p0(-1.0f, 1.0f, -1.0f);
	D3DXVECTOR3 u(1.0f, 0.0f, 0.0f);

	// Construct plane by specifying its (A, B, C, D)
	// components directly.
	float s = 1.0f / sqrtf(3);
	D3DXPLANE plane(s, s, s, -5.0f);

	// Function expects a line segment and not a ray; so we just
	// truncate our ray at p0 + 100*u to make a line segment.
	D3DXVECTOR3 isect;
	D3DXPlaneIntersectLine(&isect, &plane, &p0, &(p0 + 100*u));

	cout << isect << endl;

	return 0;
}
```

The output is:

```
(8.66025, 1, -1)
Press any key to continue . . .
```

We note $5\sqrt{3} \approx 8.66025$, so the computer result agrees with our calculation.